XX. *On the Classification of Loci.*

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Part I.—CURVES.

By a *curve* we mean a continuous one-dimensional aggregate of any sort of elements, and therefore not merely a curve in the ordinary geometrical sense, but also a singly infinite system of curves, surfaces, complexes, &c., such that one condition is sufficient to determine a finite number of them. The elements may be regarded as determined by k coordinates; and then, if these be connected by $k-1$ equations of any order, the curve is either the whole aggregate of common solutions of these equations, or, when this breaks up into algebraically distinct parts, the curve is one of these parts. It is thus convenient to employ still further the language of geometry, and to speak of such a curve as the complete or partial intersection of $k-1$ loci in flat space of k dimensions, or, as we shall sometimes say, in a k-flat. If a certain number, say h, of the equations are linear, it is evidently possible by a linear transformation to make these equations equate h of the coordinates to zero; it is then convenient to leave these coordinates out of consideration altogether, and only to regard the remaining $k-h-1$ equations between $k-h$ coordinates. In this case the curve will, therefore, be regarded as a curve in flat space of $k-h$ dimensions. And, in general, when we speak of a curve as in flat space of k dimensions, we mean that it cannot exist in flat space of $k-1$ dimensions.

The whole aggregate of linear complexes may be regarded as constituting a space of five dimensions, in which the *special* complexes, or straight lines, constitute a quadric locus. A ruled surface, or scroll, will be thus regarded as a curve lying in a quadric locus in a flat space of five dimensions. If, however, the generators of the scroll all belong to the same linear complex, the scroll must be regarded as a curve lying in a quadric locus in a flat space of *four* dimensions. And if, further, the scroll has two linear directrices, so that the generators belong to a linear congruence, then the scroll may be regarded as a curve lying on an ordinary quadric surface in three dimensions. Thus, for example, quartic scrolls having two linear directrices correspond either to quadri-quadric curves of deficiency 1 (that is, they are *elliptic* curves whose coordinates may be expressed as elliptic functions of one variable), or to the curves of deficiency 0 which are the partial intersections of a quadric and a cubic surface (that is, they are unicursal curves).

This view of ruled surfaces is made excellent use of by Voss, "Zur Theorie der windschiefen Flächen," Math. Annalen, vol. viii. p. 54.

Similar considerations apply to surfaces. By a *surface* we shall mean, in general, a continuous two-dimensional aggregate (which may also be called a *two-spread* or *two-way locus*) of any elements whatever, curves, surfaces, complexes, &c., defined by the whole or a portion of the system of solutions of $k-2$ equations among k coordinates. We shall assume that none of these equations are linear, and then shall speak of the surface as in a flat space of k dimensions. We shall in certain cases go further, and speak of an h-spread or h-way locus, viz., a locus determined by the whole or an algebraically separate portion of the system of solutions of $k-h$ equations among k coordinates; if none of these equations are linear, the h-way locus will be said to be in k dimensions. The general point of view is that of Professor CAYLEY, "On the Curves which satisfy given Conditions," Phil. Trans., Vol. 158 (1868), pp. 75–144; the methods of enumeration are those of Dr. SALMON, 'Solid Geometry,' p. 261.

Theorem A. *Every proper curve of the* n^{th} *order is in a flat space of* n *dimensions or less.* For through $n+1$ points of it we can draw a flat space of n dimensions, which must therefore contain the curve, since it meets it in a number of points greater than its order.

Thus, for example, there is no curve of the second order, in space of any number of dimensions, except a plane conic. If, therefore, a system of curves, in a plane or on any surface, is such that two curves of the system can be drawn through an arbitrary point, then the coordinates of a varying curve of the system may be represented by $x_i+2\theta y_i+\theta^2 z_i$ ($i=1, 2, 3 \ldots k$), and the envelope of the system is, in the case of plane curves, a curve having the equation $\sqrt{U}+\sqrt{V}+\sqrt{W}=0$, where U, V, W are three curves of the system; in the case of curves on a surface, it is the intersection of the surface with another having an equation of that form.*

[* Professor HENRICI has kindly written for me the following notes in elucidation of this argument:—

"In the first sentence of the paper it is stated that by a *curve* is meant any one-dimensional aggregate of any sort of elements. The definitions given are algebraical, but the reasoning later on becomes more and more geometrical.

"In this note the connexion between the algebraical definition and the geometrical reasoning will be shown in the case where the elements are plane curves of order n.

"If we suppose a curve given by its equation in point coordinates we may take the coefficients as homogeneous coordinates of the curve.

"As there are $\frac{n(n+3)}{2}$ ratios of these coefficients, it follows that all curves of order n in a plane constitute a $\frac{n(n+3)}{2}$ spread, and this will be a *flat* spread as no relation has been supposed between the coordinates.

"To determine in this spread a k-flat, $k<\frac{n(n+3)}{2}$, we have to assume a sufficient number of equations between the coordinates, or denoting by $n_1, n_2, \ldots$ curves of order n we may write down the equation of

To particularise still further, a system of conics having the characteristic $\mu=2$ must always have quadruple contact with a quartic curve; and the different species may be enumerated by studying the successive degeneration of the curve, ending with the fundamental system $\nu=1$, when it breaks up into four straight lines.

So again, there is no quadric scroll, in any number of dimensions, except the ordinary quadric surface which is in flat space of three dimensions.

A curve of the third order must be either the known skew cubic in three dimensions, or a plane cubic. Hence, if a system of curves be such that three of them can be drawn through an arbitrary point, the equation of any curve of the system is of one of the two forms—

$$U+3Vt \quad +3Wt^2+Xt^3=0,$$
$$U+\ V\,\mathrm{sn}^2u+2W\,\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u=0,$$

where t, u are parameters. Hence it is easy to write down the equations to the envelopes in the two cases, and to enumerate the distinct species.

one element in the k-flat in the form $a_1u_1+a_2u_2+\ \ldots\ +a_{k+1}u_{k+1}=0$, and take the k ratios of the a as the coordinates of a variable curve.

" For $k=2$ we get a *net* as the flat space of two dimensions or as the *plane* in this space, and for $k=1$ a pencil corresponding to the *line*.

" If, on the other hand, we assume in the k-flat $k-1$ equations between the coordinates a, there remains a singly infinite number of curves, that is according to Professor CLIFFORD a *curve* (with curves as elements), according to the usual nomenclature a series of curves.

" To determine the order of this curve we have to find the number of elements on it which satisfy a linear relation between the coordinates. In our case the condition that a curve shall pass through a given point gives such a relation, and the number of curves through a point is the *order* in question.

" Hence, if we wish to extend a theorem relating to a curve (in the ordinary sense with points as elements, but in any number of dimensions) to a proposition relating to a series of curves, or if we wish to illustrate in a plane a theorem relating to a curve in more dimensions than three, we have instead of a point on the curve to take a curve in the series, and to replace the order of the curve by the index of the series.

" The theorem that every curve of order two is a *plane* curve becomes thus—the curves in a series of index 2 belong to a *net*.

" Further, the coordinates of a point on a conic may be represented as expressions of the second degree in a variable parameter, say by $x_i+2\theta y_i+\theta^2 z_i$; where $i=1, 2, 3$, if the coordinates are taken in the plane of the conic, but if they are taken in space we have to take $i=1, 2, 3, 4$, and so on for more dimensions. The locus of these points, that is, the conic, is then given by an equation of the form

$$\sqrt{U}+\sqrt{V}+\sqrt{W}=0$$

where U, V, W are three of the points.

" If we apply this to our series we obtain the results stated in the text, viz., the coordinates of any curve of a series such that two curves pass through a given point are of the form quoted, and the equation of the envelope is of the form

$$\sqrt{U}+\sqrt{V}+\sqrt{W}=0,$$

U, V, W being three of the curves.

" Similarly, if the series is such that three pass through any point, then the series may be considered as a 'curve' of order three, and the statements made in the text follow at once from the known properties about cubic curves, which are either unicursal (twisted, or plane nodal, cubics) or they are plane curves of deficiency one."—January, 1879.]

A cubic scroll must be of the nature of the skew cubic, because it is a curve (with complexes for elements) which is obliged to lie on a quadric locus (that of the special complexes, or straight lines).

Theorem B. *A curve of order* n *in flat space of* k *dimensions* (*and no less*) *may be represented, point for point, on a curve of order* n−k+2 *in a plane.*

The proposition is obvious when $k=3$. The cone standing on a curve of order n (in ordinary space of three dimensions), and having its vertex at a point of the curve, is of order $n-1$; if then we cut this cone by a plane, we have the tortuous curve represented, point for point, on a plane curve of order $n-1$.

Now this process is applicable in general. Starting with an arbitrary point, P, of a curve in any number of dimensions, let us join this point to all the other points of the curve; we shall thus get a *cone* of order $n-1$. For any flat locus of $k-1$ dimensions drawn through the point P must meet the curve in n points, of which P is one; and therefore it must meet the cone in $n-1$ lines. Hence, if we cut this cone by such a flat $(k-1)$way locus *not* passing through P, we shall get a curve of order $n-1$ in flat space of $k-1$ dimensions, which is a point-for-point representation of the original curve. By continuing this process we may go on diminishing the order of the curve and the number of dimensions by equal quantities, until we have subtracted $k-2$ from each; when we are left with a curve of order $n-k+2$ in a plane.

The reduction may, however, be effected in one step. A flat $(k-2)$way locus may be drawn through $k-1$ arbitrary points. Suppose it to contain $k-2$ consecutive points of the curve at P, and another variable point, Q, of the curve. Such a locus will meet an arbitrary plane in one point, R. As Q then moves about on the curve, R will trace out on the plane a curve which corresponds to it, point for point. But this curve is of order $n-k+2$, for a flat $(k-1)$way locus, passing through $k-2$ consecutive points of the original curve at P, will meet that curve in $n-k+2$ other points, and therefore will meet also the locus of R in $n-k+2$ points. This locus is, therefore, of order $n-k+2$, as was to be proved.

The fixed points through which the variable $(k-2)$way locus passes need not all be united at P, but they may be any $k-2$ arbitrary points on the curve.

We will now consider some examples of this remark.

1. *Unicursal curve of order* n *in* n-*dimensional space.*

A curve of order n *in flat space of* n *dimensions* (*and no less*) *is always unicursal.*— We may prove this independently by considering a variable $(n-1)$flat which passes through $n-1$ fixed points on the curve. Its equation will be of the form $A+tA'=0$, where t is a variable parameter, and it will meet the curve in one other point, which is thus associated with a value of t.

The equations to such a curve may always be written in the form—

$$0=\left\| \begin{matrix} A, B, C \ldots K \\ B, C, D \ldots L \end{matrix} \right\| \quad \ldots \ldots \quad (1)$$

where the A, B, C . . . K, L are linear functions of the coordinates, and the number of columns is $=n$. For the $n+1$ homogeneous coordinates are proportional to rational integral functions of t of the n^{th} order. Solving these $n+1$ equations for $1, t, t^2 \ldots t^n$ we find

$$1, t, t^2 \ldots t^n = A, B, C \ldots L,$$

which is equivalent to the system written down above.

The more general system of equations—

$$0=\left\| \begin{matrix} A, B \ \ldots K \\ A', B' \ldots K' \end{matrix} \right\| \quad \ldots \ldots \quad (2)$$

where the A . . . K, A′ . . . K′ are linear functions as before, may always and easily be reduced to the former, for they are got by eliminating t from the n equations.

$$\begin{aligned} &A+tA'=0, \quad \ldots \ldots \quad (3) \\ &B+tB'=0, \\ &\vdots \\ &K+tK'=0. \end{aligned}$$

We may, however, solve these equations for the ratios of the coordinates, which will thus be expressed as rational functions of t of the n^{th} order. Solving these for $1, t, t^2 \ldots t^n$ we come back to the previous system.

The equations (3) exhibit the curve as the locus of the intersection of corresponding elements in n projective pencils.

The equation to the $(n-1)$flat which passes through the n points whose parameters are $t_1, t_2 \ldots t_n$, is easily seen to be—

$$0=\left| \begin{matrix} A, & B, & C, \ldots & L \\ 1, & t_1, & t_1^2, \ldots & t_1^n \\ 1, & t_2, & t_2^2, \ldots & t_2^n \\ \vdots & & & \\ 1, & t_n, & t_n^2, \ldots & t_n^n \end{matrix} \right|$$

But this equation is manifestly divisible by the coefficient of L, which is the product of the differences of all the t. If we write—

$$\begin{aligned} &\Sigma_1 = t_1+t_2+t_3+ \ \ldots \ +t_n, \\ &\Sigma_2 = t_1t_2+t_1t_3+t_2t_3+ \ \ldots \ +t_{n-1}t_n, \\ &\text{etc.}=\text{etc.} \\ &\Sigma_n = t_1t_2 \ \ldots \ t_n, \end{aligned}$$

then the equation is

$$0=L-K\Sigma_1+\ldots\pm B\Sigma_{n-1}\mp A\Sigma_n \quad \ldots \ldots \quad (4)$$

If we omit the suffixes of the t in this formula we obtain the equation to the osculant $(n-1)$flat at the point t. Namely (beginning at the other end), it is—

$$0=\mathrm{A}t^n-n\mathrm{B}t^{n-1}+\tfrac{1}{2}n(n-1)\mathrm{C}t^{n-2}-\ldots\pm n\mathrm{K}t\mp\mathrm{L} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and we see at once that *the class of such a curve is always equal to its order.*

We thus obtain a very useful representation (*Abbildung*) of the points of the n-dimensional space by means of groups of n points on such a unicursal curve, namely, each point in the space is represented by the points of contact of the n osculant $(n-1)$flats which pass through it. The use of such a representation of ordinary three-dimensional space by means of a skew cubic was pointed out by Dr. HIRST, and the corresponding representation of a plane by means of a conic has been used by M. DARBOUX ('Sur une classe remarquable de courbes et de surfaces algébriques,' Paris, 1873, Note II., p. 183), and by me ("On the Transformation of Elliptic Functions," Proc. Lond. Math. Soc., vol. vii. (1875), pp. 25–38 and 225–233). It may be worth while to mention that an extension to all space of the theory of the in-and-circumscribed polygon may be obtained by this means.

A curve of this kind determines also a dualistic correspondence in the space of n dimensions. Through every point may be drawn n osculant $(n-1)$flats, and through their points of contact another $(n-1)$flat, which shall be called the *polar* of the point. If the point moves along a straight line its polar will pass through a fixed $(n-2)$flat, the *polar* of the line. And generally if the point lies in any k flat the polar will pass through a fixed $(n-k-1)$flat.

When $n=2$ we have the ordinary system of polar reciprocals in regard to a plane conic. When $n=3$ we have that system in regard to a skew cubic which is described by SCHRÖTER, 'Crelle,' vol. lxv. p. 39. These two systems are typical respectively of the cases in which n is even and odd. When n is even, the relation between the coordinates of two points, which expresses that each lies in the polar of the other, is a symmetrical one; consequently those points which lie in their own polars are points on a certain quadric locus, and the system is merely that of the poles and polars in regard to this quadric locus upon which the curve lies. The equation to this locus is at once obtained by equating to zero the quadrinvariant of the form $(1, t)^n$ which occurs in the equation (5) of the osculant $(n-1)$flat, namely, it is

$$0=\mathrm{AL}-n\mathrm{BK}+\tfrac{1}{2}n(n-1)\mathrm{CH}-\text{etc.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

To prove this, observe that if in the equation (5) we substitute the coordinates of any point p, the values of t which satisfy the equation are the parameters of the points of contact of the osculant $(n-1)$flats which pass through the point. If $t_1, t_2, \ldots t_n$ be these values, the equation (4) represents the $(n-1)$flat which passes through the points of contact, that is to say, the polar of the point. Now if we denote by A′, B′, . . . the results of substituting the coordinates of the point p in A, B, . . . then we shall have—

$$A'\Sigma_1 = nB', \quad \ldots \quad (7)$$
$$A'\Sigma_2 = \tfrac{1}{2}n(n-1)C',$$
$$\vdots$$
$$A'\Sigma_n = L'$$

so that, when n is even the equation of the polar is—

$$0 = AL' + A'L - n(BK' + B'K) + \tfrac{1}{2}n(n-1)(C'H + C'H) - \text{etc.} \quad \ldots \quad (8)$$

that is, it is simply the polar of the point in regard to the quadric (6).

It is to be observed that the quadric is completely determined when the curve is given. I reserve the question of the conditions to which the curve is subject when the quadric locus is given, or, say, the discussion of the problem to represent the relation of poles and polars in regard to a quadric locus (in space of an even number of dimensions) by means of a unicursal curve.

But when n is odd, the last term of equation (4) is negative, and the equation of the polar is—

$$0 = AL' - A'L - n(BK' - B'K) + \tfrac{1}{2}n(n-1)(CH' - C'H) - \text{etc.} \quad \ldots \quad (9)$$

that is, it is skew symmetrical, and *every point lies upon its polar.* It is convenient to use the term *co-flat* for $n+1$ points, which are in the same $(n-1)$flat; with this nomenclature we may say that *when* n *is odd every point is co-flat with the* n *points of contact of the osculant* (n$-$1)*flats, which can be drawn through it.* This will be recognised as an extension of the property of a skew cubic, that every point in space is co-planar with the points of contact of the three osculating planes which can be drawn through it.

A case of this skew symmetrical relation is given by any arbitrary state of motion of the whole space as a rigid body, the relation between two points being that the line joining them moves perpendicularly to itself. The polar of any point is an $(n-1)$flat drawn through it perpendicular to the direction of its motion. When n is even there is always one point which remains at rest, and all the polars pass through this point. Thus the general motion of a solid in an even number of dimensions always depends in this simple way on the motion in one dimension less. In an odd number of dimensions, however, every point moves in the general case; but if any point is at rest, then all the points in a certain straight line are at rest.

Besides its order and class, a curve has, in general, characteristic numbers intermediate to these, which may be called its first rank, second rank, etc. The first rank is the order of the locus traced out by straight lines through two consecutive points of the curve; the second rank, of that traced out by planes through three consecutive points; and generally the k^{th} rank is the order of the $(k+1)$wide locus traced out by k-flats through $k+1$ consecutive points. For the curve just considered these numbers are $2(n-1)$, $3(n-2)$, . . . $(k+1)(n-k)$; it is convenient to derive them from the corresponding numbers for its projection, the unicursal curve of order n in $n-1$ dimensions, to which we now proceed.

2. *Unicursal curve of order* n *in* n−1 *dimensions.*

Every curve of order n *in flat space of* n−1 *dimensions is either unicursal or elliptic.* For it may be represented point-for-point on a plane cubic.

We shall treat these two cases in succession. They are exemplified by the two species of quartics in ordinary tri-dimensional space.

The coordinates of a point on the unicursal curve are proportional to rational integral functions of a parameter t. This representation may be simplified in a manner due to ROSANES, 'Crelle,' vol. lxxv. p. 166. We have n binary quantics of order n; now these may be linearly combined in n different ways so as to produce a perfect n^{th} power. Hence the original quantics may be expressed each as a linear function of the n^{th} powers of the same n linear quantics. Thus, for example, three binary cubics may be simultaneously reduced to the forms.

$$\begin{gathered} au^3+b\ v^3+c\ w^3, \\ a'u^3+b'\ v^3+c'\ w^3, \\ a''u^3+b''v^3+c''w^3, \end{gathered}$$

where $u+v+w=0$ identically. If the x, y, z of a point in a plane are respectively proportional to these cubics, we may, by solving the equations for u^3, v^3, w^3, obtain three linear functions X, Y, Z of the coordinates, which are respectively proportional to u^3, v^3, w^3. Transforming them to the new triangle whose sides are $X=0$, $Y=0$, $Z=0$ we must have the equation of a unicursal cubic expressed in the form

$$X^{\frac{1}{3}}+Y^{\frac{1}{3}}+Z^{\frac{1}{3}}=0.$$

It is clear that the lines $X=0$, $Y=0$, $Z=0$ are tangents at the three points of inflexion.

In general, let the n quantics be

$$\begin{gathered} a_0+na_1t+\ldots+a_nt^n, \\ b_0+nb_1t+\ldots+b_nt^n, \\ \vdots \\ h_0+nh_1t+\ldots+h_nt^n, \end{gathered}$$

then the linear quantics u, v, w . . . are the factors of

$$\begin{vmatrix} t^n, & -nt^{n-1}, & \frac{1}{2}n(n-1)t^{n-2}, & \ldots t \\ a_0, & a_1, & a_2, & \ldots a_n \\ b_0, & b_1, & b_2, & \ldots b_n \\ \vdots & \vdots & \vdots & \vdots \\ h_0, & h_1, & h_2, & \ldots h_n \end{vmatrix}$$

Since there are $n-2$ identical relations between n linear quantics, the $n-2$ equations of the unicursal curve may be written in the form

$$\left\| \begin{array}{llll} X_1^{\frac{1}{n}}, & X_2^{\frac{1}{n}}, & \ldots & X_n^{\frac{1}{n}} \\ \alpha_1, & \alpha_2, & \ldots & \alpha_n \\ \beta_1, & \beta_2, & \ldots & \beta_n \end{array} \right\| = 0\,;$$

it is evident that the equations $X_1, X_2, \cdots X_n=0$ represent stationary osculant $(n-2)$flats, that is to say, $(n-2)$flats which pass through n consecutive points of the curve.

The properties of this curve may be very conveniently studied by regarding it as a projection of the curve considered in the last section. If all the points of that curve be joined to a point O, not situated upon it, the joining lines will form a cone of order n; and on cutting this cone by an $(n-1)$flat we shall obtain the curve now under discussion.

The n points of superosculation, whose existence has just been proved, are then clearly the projections of the points of contact of osculant $(n-1)$flats to the full-skew curve drawn through the point O. It follows that *when* n *is odd, these* n *points of superosculation are on the same* (n$-$2)*flat;* but when n is even this is not the case, unless the point O lies on the quadric locus associated with the full-skew curve, in which case we have a special variety of the projection. Thus the three points of inflexion of a nodal cubic are in one straight line; but a unicursal skew quartic in ordinary space has not in general the property that the points of contact of its four stationary osculating planes are in one plane. The property established above for the full-skew curve shows that this will be the case if the four points form an equianharmonic system, or if the quadrinvariant of the quartic which determines them is equal to zero. And generally when n is even, the n points of superosculation will be co-flat if, and only if, the quantic in t which determines them has its quadrinvariant zero.

By using the values of the coordinates of a variable point of the curve expressed in terms of a parameter t, we may obtain an expression of this quadrinvariant and also of its product by the discriminant in terms of the roots of the quantic. Let $\alpha_1, \alpha_2, \ldots \alpha_n$ be the values of t which belong to the points of superosculation, and $x_1, x_2, \ldots x_n$ the coordinates of a variable point on the curve. Then we may write

$$x_i=(t-\alpha_i)^n, \qquad i=1, 2, \ldots n,$$

and the coordinates of the point α_i are $(\alpha_i-\alpha_1)^n, (\alpha_i-\alpha_2)^n, \ldots (\alpha_i-\alpha_n)^n$. If for shortness we write $(h\,k)$ instead of $\alpha_h-\alpha_k$, then the condition that the n points shall be co-flat is

$$0=\begin{vmatrix} 0\ , & (12)^n, & (13)^n, & \ldots & (1n)^n \\ (21)^n, & 0\ , & (23)^n, & \ldots & (2n)^n \\ (31)^n, & (32)^n, & 0\ , & \ldots & (3n)^n \\ \vdots & \vdots & \vdots & & \vdots \\ (n1)^n, & (n2)^n, & (n3)^n, & \ldots & 0 \end{vmatrix}.$$

This is obviously always satisfied if n is odd, for then the determinant is skew symmetrical, and being of odd order it necessarily vanishes. If, however, n is even, the determinant is a symmetrical function of the roots which vanishes when any two of them are equal; and consequently it must contain as a factor the product of the squares of their differences. Now the determinant is of the order $2n$ in each root, and the discriminant is of order $2(n-1)$; therefore the remaining factor is of order 2 in each root, and being a symmetrical invariant must be a function of the squares of their differences. It can therefore be no other than $\Sigma(\alpha_1-\alpha_2)^2(\alpha_3-\alpha_4)^2 \ldots (\alpha_{n-1}-\alpha_n)^2$; this is, to a factor *près*, equal to the quadrinvariant of the form $(t-\alpha_1)(t-\alpha_2)\ldots(t-\alpha_n)$.

The equation to the $(n-2)$flat passing through two consecutive points of the curve at t, and through $n-3$ other points $p\ q \ldots u$, is clearly

$$0=\begin{vmatrix} x\ dx\ p\ q \ldots y \\ 1\ \ 2\ \ 3\ 4 \ldots n \end{vmatrix}$$

where the y are current coordinates, and the determinant is expressed in umbral notation. Writing in this for x_i, $(t-\alpha_i)^n$, and for dx_i, $n(t-\alpha_i)^{n-1}dt$, we may observe that the determinant

$$\begin{vmatrix} (t-\alpha_1)^n\ , & (t-\alpha_2)^n \\ (t-\alpha_1)^{n-1}, & (t-\alpha_2)^{n-1} \end{vmatrix} = (\alpha_2-\alpha_1)(t-\alpha_1)^{n-1}(t-\alpha_2)^{n-1},$$

so that the equation is of order $2(n-1)$ in t. It thence follows that $2(n-1)$ different $(n-2)$flats may be drawn through $n-2$ arbitrary points to touch the curve; or that the developable traced out by the tangent lines is of the order $2(n-1)$.

Similarly, from the value of the determinant

$$\begin{vmatrix} (t-\alpha_1)^n\ , & (t-\alpha_2)^n\ , & \ldots & (t-\alpha_{k+1})^n \\ (t-\alpha_1)^{n-1}, & (t-\alpha_2)^{n-1}, & \ldots & (t-\alpha_{k+1})^{n-1} \\ \vdots & \vdots & & \vdots \\ (t-\alpha_1)^{n-k}, & (t-\alpha_2)^{n-k}, & \ldots & (t-\alpha_{k+1})^{n-k} \end{vmatrix}$$

which is equal to the product of the differences of $\alpha_1, \alpha_2, \ldots \alpha_{k+1}$ multiplied by

$$\{(t-\alpha_1)(t-\alpha_2)\ \ldots\ (t-\alpha_{k+1})\}^{n-k},$$

we may conclude that the number of $(n-2)$flats which can be drawn through k consecutive points of the curve and through $n-k$ other arbitrary points is $(k+1)(n-k)$; or that the k-wide locus which is traced out by $(k-1)$flats passing through k consecutive points is of the order $(k+1)(n-k)$. For the equation of an $(n-2)$flat passing through k consecutive points is clearly

$$0=\left|\begin{matrix} x\ dx\ d^2x\ \ldots\ d^{k-1}x\ p\ q\ \ldots\ y \\ 1\ \ 2\ \ 3\ \ldots\ k, \qquad \ldots\ n \end{matrix}\right|$$

where we must substitute for the x_i, dx_i, d^2x_i, etc., the descending powers of $t-a_i$ beginning at the n^{th}. Making k equal to $n-1$ we obtain the equation of the osculant $(n-2)$ flat at any point of the curve; it is

$$0=\frac{P_1y_1}{(t-a_1)^2}+\frac{P_2y_2}{(t-a_2)^2}+\ldots+\frac{P_ny_n}{(t-a_n)^2},$$

where P_i = product of the differences of all the a except a_i. Thus the class of the curve is $2(n-1)$.

3. *Unicursal curve of order* n *in* n—k *dimensions.*

The characteristic numbers belonging to this curve may at once be obtained by regarding it as a projection of the full-skew curve. The number of ranks is $n-k-2$, and the numerical values of them are respectively $2(n-1)$, $3(n-2)$, ... $k(n-k+1)$; the class is $(k+1)(n-k)$; and the number of points of superosculation is $(k+2)(n-k-1)$. For example, the unicursal quintic in three dimensions is of rank 2.4, =8, and of class 3.3, =9, and it has 4.2, =8 superosculant planes.

Convenient forms of the equations may be got by eliminating some of the variables from the equations of the full-skew curve; but care must be taken to select these variables so that the resulting system is sufficiently general.

4. *Elliptic (or bicursal) curve of order* n *in* n—1 *dimensions.*

We have proved already that a curve of order n in $n-1$ dimensions can be represented, point for point, on a plane cubic. If, therefore, it is not unicursal, its coordinates can be expressed in terms of elliptic functions of a single parameter. Now, it follows from the investigations of CLEBSCH, 'Crelle,' vol. lxiv. (1864), pp. 210–270, that if n points of the curve are co-flat, the sum of their parameters will differ from a certain constant by a sum of integer multiples of the two periods of the elliptic function. Let the periods be ω and ω', then if t_1, t_2, ... t_n are the parameters of the points,

$$t_1+t_2+\ldots+t_n=c+a\omega+b\omega',$$

where c is a constant, and a, b are integers. To find the points of superosculation, we must suppose the n points to become identical, or the t, still satisfying this equation, to become equal. We thus obtain

$$nt = c + a\omega + b\omega$$

$$t = \frac{c}{n} + \frac{a}{n}\omega + \frac{b}{n}\omega',$$

and values of t, representing distinct points, will be got by giving to the numbers a, b the values $0, 1, \ldots n-1$ independently. Hence *there are* n^2 *points of superosculation.*

Thus a plane cubic has nine inflexional tangents, and a quadri-quadric curve has sixteen superosculant planes.

Propositions hold good in general in regard to the grouping of these points, which are analogous to those which relate to the inflexions of a cubic. Thus, *an* $(n-2)$*flat drawn through* $n-1$ *of them will always pass either through another besides, or through the tangent line at one of the* $n-1$. This is obvious from the values already given for the parameters of points of superosculation.

Through any point of the curve can be drawn $(n-1)^2$ osculant $(n-2)$flats. This is proved in the same way as the preceding proposition, which is, in fact, the projection of it; for if through the given point we draw a cone containing the curve, and cut it by an $(n-2)$flat, the section will be an elliptic curve of order $n-1$ in $n-2$ dimensions, and the projections of the points whose osculant $(n-2)$flats pass through the given point will be points of superosculation on the projected curve. Hence, also, the lines joining the given point to the points of contact are grouped in respect of co-flatness in the same way as the points of superosculation in the curve of next lower order.

More generally, through k given points of the curve there can be drawn $(n-k)^2$ $(n-2)$flats which have $(n-k)$ pointic contact with the curve. If $u_1, u_2, \ldots u_k$ are the parameters of the k given points, those of the required points are given by

$$u = \frac{1}{n-k}(u_1 + u_2 + \ldots u_k + a\omega + b\omega'),$$

where the integers a, b may take independently the values $0, 1, \ldots n-k-1$.

From these results we may now determine the various ranks and the class of the curve. Suppose that we know the number of $(n-2)$flats which can be drawn through $n-2$ arbitrary points in space—or, which is the same thing, through an arbitrary $(n-3)$flat P—to touch a certain curve. Then, if the arbitrary $(n-3)$flat meets the curve in any point, *two* of these will coincide at that point. For taking an $(n-4)$flat in the $(n-3)$flat, and joining it to all the points of the curve by $(n-3)$flats, we may cut this figure by a plane or 2 flat. Every $(n-3)$flat will cut this plane in a single point. The problem is then reduced to drawing tangents from a point (viz., the intersection of P by the plane) to a plane curve; and we know that when this point lies on the curve, two of the tangents coincide at it.

In general, a certain number of $(n-2)$flats can be drawn through an arbitrary $(n-k-1)$flat to have k-pointic contact with a given curve; this number is, in fact, the $(k-1)^{\text{th}}$ rank of the curve. If the arbitrary $(n-k-1)$flat meets the curve at any point, then k of these $(n-2)$flats coincide at that point. For we may project the whole figure from an $(n-k-2)$flat lying in the $(n-k-1)$flat on to a k flat. The problem is then reduced to drawing $(k-1)$flats through a given point to have k-pointic contact with a curve in k dimensions. Now we know, from the example of the full-skew curve, that, when the point lies on the curve, k of these coincide at the point.

If the arbitrary $(n-k-1)$flat meet the curve in more points than one, k of the osculants will coincide at each of them; and this result is not affected by the union of the points into one. In particular, if it meet the curve in $n-k$ coincident points, the number of osculants which there coincide is $k(n-k)$.

Applying now these general considerations to the elliptic curve, we find at once that the $(k-1)^{\text{th}}$ rank of it is nk. For we must add to the number $k(n-k)$, just obtained, the number, k^2, given by the theory of elliptic functions for the $(n-2)$flats drawn through $n-k$ consecutive points of the curve to have k-pointic contact elsewhere. In particular, the class of the curve is $n(n-1)$; we have observed already that the number of superosculants is n^2.

Thus, a plane cubic is of order 3, class 6, and has 9 inflexions; a quadri-quadric is of order 4, rank 8, class 12, and has 16 superosculant planes. We learn, moreover, that a quintic curve in four dimensions, when not unicursal, is of first rank 10, second rank 15, class 20, with 25 points of superosculation. Hence a quintic in three dimensions, with five apparent dps., is of rank 10, class 15, and has 20 superosculant planes; this follows by projection from the former case.

A curve of this kind, viz., an elliptic curve of order n in an $(n-1)$flat has its coordinates $x_1, x_2 \ldots x_n$ determined by the equations

$$x_1, x_2, \ldots x_n = 1, t, t', t^2, tt' \ldots$$

(the last term on the right being $t^{(\frac{1}{2}n-1)}t'$ or else $t^{\frac{1}{2}n}$, according as n is odd or even), where $t = \text{sn}^2(u + i\text{K}')$ and

$$t' = \frac{dt}{du} = 2\,\text{sn}(u+i\text{K}')\text{cn}(u+i\text{K}')\text{dn}(u+i\text{K}') = \sqrt{2t(1-t)(1-k^2t)}.$$

[If n is even, we may write $t = \text{sn}^2 u$ instead of $\text{sn}^2(u+i\text{K}')$.] The condition for n points $u_1, u_2 \ldots u_n$ to be co-flat is then

$$u_1 + u_2 + \ldots + u_n = 0.$$

See LINDEMANN; CLEBSCH'S 'Lectures on Geometry,' vol. ii.

[*Theory of Derived Points on an Elliptic* (*or Bicursal*) *Curve.*

SYLVESTER's theory of derived points on a plane cubic is as follows :—Starting from any given point on the curve, we may construct its *tangential*, or point where the tangent at the original point meets the curve again; similarly we may construct the tangential of the tangential, or second tangential, and so on. By joining any two non-consecutive points on this series, we can find their *residual*, the point where the joining line meets the curve again. In this way we obtain an infinite group of points derived from (and including) the original point, such that the line joining any two of them is either tangent at one of these or passes through a third point of the group. It is to be observed that all points on the curve uniquely derived from the given point by any geometrical process (*e.g.*, the point where the conic of five-pointic contact meets the curve again, the point where cubics of eight-pointic contact meet the curve again, &c.) are included in the group.

The coordinates of any derived point may be expressed rationally in terms of the coordinates of the original point, and the order of the functions to which they are proportional is always a square number. Thus the three coordinates of the tangential are proportional to quartic functions of the coordinates of the original. If the square root of the order of these functions be called the order of the derived point, then we have the theorem that when three derived points are in a straight line, the order of one of them is equal to the sum of the orders of the other two. It is observed that there is no derived point whose order is divisible by 3. By help of this observation it is easy to make out a scheme of the orders; for when we join two points, the order of their residual must be the sum or the difference of the orders of the points, and one or the other of these is always divisible by 3.

This theory is really a geometric representation of the multiplication of elliptic functions. The coordinates x_1, x_2, x_3 of any point on the cubic curve may by proper choice of axes be made proportional to elliptic functions of a parameter u, so that

$$x_1 : x_2 : x_3 = 1 : \mathrm{sn}^2(u+i\mathrm{K}') : \mathrm{sn}(u+i\mathrm{K}')\,\mathrm{cn}(u+i\mathrm{K}')\,\mathrm{dn}(u+i\mathrm{K}').$$

This being so, if u, v, w are parameters of three points in a straight line, we shall have $u+v+w=0$. If v be the tangential of u, the three points u, u, v are in a straight line, and $2u+v=0$, or $v=-2u$. Hence the series of tangentials has for parameters

$$u,\ -2u,\ +4u,\ -8u,\ \&\text{c.}:$$

and in general the parameter of any derived point is of the form nu, where n is a positive or negative integer. The number n, taken positively, coincides with what was called the order of the derived point. For the elliptic functions of nu are of the order n^2 in the elliptic functions of u.

In this way all points of the theory are explained, excepting the fact that no derived point has its order divisible by 3.

Moreover, we see at once that the theory can be extended to other curves of deficiency 1; as, for example, the quadri-quadric curve. Starting with any point on this curve, we may find the point where the osculating plane at that point meets the curve again; then repeat the process with the point so found, and so on. The plane joining any three of these points will meet the curve in another derived point, or else touch it at one of the three points. The plane drawn through one derived point to touch the curve at another derived point will meet it again in a derived point, *or* touch at the first point, *or* osculate at the second. The coordinates of any derived point are of the order n^2 in those of the original point, where $\pm n$ may be called the order of the derived point. In this case the order of no derived point is divisible by 2.

I was desirous of finding a similar representation of the multiplication of hyper-elliptic and Abelian functions; and therefore sought for cases in which derived elements might be found on *curves* (in the sense explained in the beginning of this paper) of deficiency greater than 1. For this purpose I considered scrolls. Taking an arbitrary generator on a quartic scroll having two linear directrices, we may draw a one-sheeted hyperboloid through three consecutive generators at that place; this will meet the quartic scroll in one other generator, which is thus uniquely derived from the given one. Similarly on a quintic scroll contained in a linear complex, the two tractors of four consecutive generators meet the scroll in two other points lying on a generator. And on a sextic scroll not contained in a linear complex, the linear complex having five-line contact at a given generator (containing five consecutive generators) will contain one other generator of the scroll. In these three cases, then, from any three, four, or five generators we may uniquely derive a fourth, fifth, or sixth generator respectively; and the whole theory of derived elements may be applied to the generators of these scrolls.

Unfortunately, however, each of the scrolls considered is at most of deficiency 1, so that we merely get more illustrations of the multiplication of elliptic functions. And it may be shown, in general, that a *curve* on which such a theory of derived points is possible, is at most of deficiency 1.

Suppose that it has no singular points, and that $k-1$ points on it being given, there is uniquely determined one other point. If this is effected (as in the above examples) by drawing a flat space through the $k-1$ points, which meets the curve in one other point, then it must be of the order k. Moreover, it must be in a flat space of so many dimensions that the flat of one dimension less is determined by $k-1$ points. Now a $(k-2)$flat is determined by $k-1$ points; therefore, the curve is in a $(k-1)$flat.

Thus the impossibility of extending the theory of derivation to curves of deficiency greater than unity is equivalent to the proposition that a curve of order k in $k-1$ dimensions is at most of deficiency 1. This failure was the starting point of the present paper.

It remains to explain why, in the group of numbers expressing the orders of the derived points, only certain forms present themselves. Let that number which, with $k-1$ other numbers, makes up zero, be called the *residual* of those numbers; it is, in fact, their sum taken negatively. Then the process of forming the group is to start from unity, and add the residual of every $k-1$ numbers of the group, repetitions being allowed. I say that by this process we shall only get numbers of the form $mk+1$. For let m_1k+1, m_2k+1, &c. be $k-1$ such numbers, then their residual is $-(m_1+m_2+\ldots)k-k-1$, which is a number of the same form. Now as unity, with which we start, is of this form, it follows that all the numbers of the group must be of the form $mk+1$.—January, 1879.]

CURVES OF DEFICIENCY p.

Theorems relating to Abelian Functions.

It will be convenient to put together shortly those propositions relating to the application of Abelian functions to curves which will be wanted in the sequel.

The aggregate of the real and imaginary points on a curve constitutes a two-way spread, or surface, which may be transformed, by stretching without tearing, into the surface of a body with p holes in it. On this surface there are $2p$ distinct closed curves which cannot without breaking be shrunk into a point, namely, one round each hole, and one through each hole. Any other irreducible circuit must be made up of combinations of these.

If any rational function of the coordinates be integrated from one point to another along the curve-spread, the value of the integral will depend upon the path of the integration. If the integral becomes infinite at any points, it may be altered in value by making the path go round one or more of these; but in any case it may be altered by incorporating into the path any of the $2p$ closed circuits just mentioned. It is found that there are p distinct rational functions of the coordinates whose integrals do not become infinite at any point of the curve-spread. Any other integral which is everywhere finite must be a linear combination of these. Of such linear combinations it is convenient to take a certain set as the *normal* set; they are so chosen that each of them, when integrated along a closed path which goes *round* a hole, gives zero for all the holes but one, and πi for that one; thus, the p integrals, which we may call $u_1, u_2, u_3, \ldots u_p$, are associated one by one with the p holes $1, 2, \ldots p$. If they are integrated along a closed curve passing *through* the hole h, let the values be called a_{h1}, a_{h2}, a_{hp}; then it is found that $a_{hk}=a_{kh}$, or the integral of u_h through the hole k is equal to the integral of u_k through the hole h.

If we now take all the integrals from a point x to a point y along the same path, and if $u_1, u_2, \ldots u_p$ are the set of values for one such path, and $U_1, U_2, \ldots U_p$ for another path, then we must have

$$
\begin{aligned}
U_1 &= u_1 + m_1\pi i + q_1 a_{11} + q_2 a_{12} + \ldots + q_p a_{1p},\\
U_2 &= u_2 + m_2\pi i + q_2 a_2 + q_2 a_{22} + \ldots + q_p a_{2p},\\
&\vdots\\
U_p &= u_p + m_p\pi i + q_1 a_{p1} + q_2 a_{p2} + \ldots + q_p a_{pp},
\end{aligned}
$$

where the numbers m, q are integers; namely, m_h is the additional number of times the new path has gone round the hole h, and q_h is the additional number of times it has gone through that hole. We shall write these equations thus

$$U_1, U_2, \ldots U_v \equiv u_1, u_2 \ldots u_p \ (\text{mod. } \pi i, a),$$

and shall say that the quantities U are *congruent* to the quantities u in respect of the periods πi, a.

Suppose now that the curve has no actual nodes, and that a locus of any order intersects it in the points $x_1, x_2, \ldots x_m$. Then, if another locus of the same order intersects it in the points $y_1, y_2, \ldots y_m$, and we take any one of the integrals, say u, from x_1 to y_1, from x_2 to y_2, ... from x_m to y_m, the sum of these results will be congruent to zero. That is to say

$$\Sigma \int_x^y du_h \equiv 0.$$

Here the Σ refers to the suffixes of the x and y, not to h. There are p such equations. This is ABEL'S Theorem.

When the curve is in a k-flat and of the order n, we shall use this theorem chiefly for its n intersections with a $(k-1)$flat. If we regard the lower limits of the integrals (the points x) as fixed, the integrals for any point y may be regarded as parameters belonging to that point, and then ABEL'S Theorem gives us p equations between the parameters of n points which lie on a $(k-1)$flat. The truth of these equations is *necessary* to the points lying on a $(k-1)$flat, but it may not be sufficient. Thus in a bicircular quartic curve, $p=1$, we have one equation to express that four points are in a straight line, and if the points are collinear the equation is true. But it does not follow from the equation that the points are collinear; in fact, the equation holds equally good if the points are in a circle.

If the sums of the parameters of p points are given, that is, if we have the p equations—

$$\left(\int^{x_1} + \int^{x} + \ldots + \int^{x_p}\right) du_h = v_h \qquad \left[h=1, 2, \ldots p\right],$$

the v_h being arbitrary constants, and the lower limits of the integrals being supposed constant; then the upper limits $x_1, x_2, \ldots x_p$ may be expressed in terms of the quantities v_h—namely, they are the roots of an equation of degree p whose coefficients are products of ϑ-functions of the v. If

$$\phi(m_1, m_2, \ldots m_p) = \Sigma m_h m_k a_{hk} + 2\Sigma m_h v_h,$$

then

$$\vartheta(v_1, v_2, \ldots v_p)=\Sigma^p e^{\phi(m)},$$

the Σ^p indicating that each of the p integers $m_1, m_2, \ldots m_p$ is to take all integral values positive and negative. When the lower limits are so chosen that the sum of the parameters is zero for the complete intersection by any locus, this ϑ-function has remarkable properties. If we sum each of the parameters for any $p-1$ points on the curve, the ϑ-function whose arguments are these sums will vanish. That is if

$$v_h=\left(\int^{x_1}+\int^{x_2}+\ldots\int^{x_{p-1}}\right)du_h,$$

then $\vartheta(v_1 v_2 \ldots v_p)=0$. If these sums are taken for any $p-2$ points, not only will the ϑ vanish, but also its differential coefficient in regard to any one of the points. And generally, if we take for the v the sums of the parameters for $p-r$ points, the ϑ and its first $r-1$ differential coefficients in regard to any of the points will vanish.

Conversely, if the p quantities v are such that $\vartheta(v)$ and its first $r-1$ differential coefficients vanish, then it is possible to find $p-r$ points x such that

$$\left(\int^{x_1}+\int^{x_2}+\ldots+\int^{x_{p-r}}\right)du_h=v_h.$$

Although here the number of equations is greater by r than the number of unknown quantities, yet it is possible to satisfy them all in virtue of the relations existing between them.

Relation between the Order and Deficiency of a Curve.

We shall now apply these theorems to the study of curves existing in k dimensions, of the order n and deficiency p. A $(k-1)$flat cuts such a curve in n points, such that the sum of each of the p parameters, for the n points, is zero. But a $(k-1)$flat is determined by k points; so that, k arbitrary points being chosen on the curve, it is always possible to find $n-k$ other points, so that the sum of each parameter for the whole n points shall be zero. Let then $-v_1, -v_2, \ldots -v_p$ be the sums of the parameters for the given k points; then to find the remaining $n-k$ points we have the p equations

$$\left(\int^{x_1}+\int^{x_2}+\ldots+\int^{x_{n-k}}\right)du_h\equiv v_h.$$

If p is not greater than $n-k$, we know that these equations can be solved, although the solution may be indeterminate. But if $p>n-k$, the equations cannot be solved unless certain conditions are satisfied by the v. Let $p-n+k=r$, then r conditions must be satisfied; namely the v must be sums of the parameters of not more than $p-r$ (or $n-k$) points. But they are sums of the parameters of k points; therefore k is not greater than $n-k$, or $2k$ is not greater than n. We have proved then that

If p>n−k, *then* 2k *is not greater than* n.

Conversely, *if* k>$\frac{1}{2}$n, p *is at most equal to* n−k.

We may also state the proposition in this way. *A curve of order* n *and deficiency* p, *not greater than* $\frac{1}{2}$n, *can at most exist in* n−p *dimensions.*

[It appears, therefore, that the theorems at the beginning of the paper may be extended, and that in n dimensions we have the curve of order n which is unicursal, the curve of order $n+1$, and deficiency at most 1, of order $n+2$, and deficiency at most 2, and so on till we come to the order $2n$, which is the first case of exception, and may have deficiency $n+1$. This curve is the natural geometric representation of the general Abelian functions, its multiple tangent flats playing the same part as the double tangents of the quartic curve in RIEMANN'S beautiful paper on the case $p=3$. H. WEBER has noticed that in four dimensions this curve is the complete intersection of three quadric loci.—January, 1879.]